

2.7 Differentiation

Derivatives (general)

Power	$\frac{d}{dx}(u^n) = nu^{n-1} \frac{du}{dx}$	(2.292)	n	power index
Product	$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$	(2.293)	u, v	functions of x
Quotient	$\frac{d}{dx} \left(\frac{u}{v} \right) = \frac{1}{v} \frac{du}{dx} - \frac{u}{v^2} \frac{dv}{dx}$	(2.294)		
Function of a function ^a	$\frac{d}{dx}[f(u)] = \frac{d}{du}[f(u)] \cdot \frac{du}{dx}$	(2.295)	$f(u)$	function of $u(x)$
Leibniz theorem	$\frac{d^n}{dx^n}[uv] = \binom{n}{0} v \frac{d^n u}{dx^n} + \binom{n}{1} \frac{dv}{dx} \frac{d^{n-1} u}{dx^{n-1}} + \cdots$ $+ \binom{n}{k} \frac{d^k v}{dx^k} \frac{d^{n-k} u}{dx^{n-k}} + \cdots + \binom{n}{n} u \frac{d^n v}{dx^n}$	(2.296)	$\binom{n}{k}$	binomial coefficient
Differentiation under the integral sign	$\frac{d}{dq} \left[\int_p^q f(x) dx \right] = f(q) \quad (p \text{ constant})$	(2.297)		
	$\frac{d}{dp} \left[\int_p^q f(x) dx \right] = -f(p) \quad (q \text{ constant})$	(2.298)		
General integral	$\frac{d}{dx} \left[\int_{u(x)}^{v(x)} f(t) dt \right] = f(v) \frac{dv}{dx} - f(u) \frac{du}{dx}$	(2.299)		
Logarithm	$\frac{d}{dx}(\log_b ax) = (x \ln b)^{-1}$	(2.300)	b a	log base constant
Exponential	$\frac{d}{dx}(e^{ax}) = ae^{ax}$	(2.301)		
Inverse functions	$\frac{dx}{dy} = \left(\frac{dy}{dx} \right)^{-1}$	(2.302)		
	$\frac{d^2 x}{dy^2} = -\frac{d^2 y}{dx^2} \left(\frac{dy}{dx} \right)^{-3}$	(2.303)		
	$\frac{d^3 x}{dy^3} = \left[3 \left(\frac{d^2 y}{dx^2} \right)^2 - \frac{dy}{dx} \frac{d^3 y}{dx^3} \right] \left(\frac{dy}{dx} \right)^{-5}$	(2.304)		

^aThe “chain rule.”



Trigonometric derivatives^a

$\frac{d}{dx}(\sin ax) = a \cos ax$	(2.305)	$\frac{d}{dx}(\cos ax) = -a \sin ax$	(2.306)
$\frac{d}{dx}(\tan ax) = a \sec^2 ax$	(2.307)	$\frac{d}{dx}(\csc ax) = -a \csc ax \cdot \cot ax$	(2.308)
$\frac{d}{dx}(\sec ax) = a \sec ax \cdot \tan ax$	(2.309)	$\frac{d}{dx}(\cot ax) = -a \csc^2 ax$	(2.310)
$\frac{d}{dx}(\arcsin ax) = a(1 - a^2 x^2)^{-1/2}$	(2.311)	$\frac{d}{dx}(\arccos ax) = -a(1 - a^2 x^2)^{-1/2}$	(2.312)
$\frac{d}{dx}(\arctan ax) = a(1 + a^2 x^2)^{-1}$	(2.313)	$\frac{d}{dx}(\operatorname{arccsc} ax) = -\frac{a}{ ax }(a^2 x^2 - 1)^{-1/2}$	(2.314)
$\frac{d}{dx}(\operatorname{arcsec} ax) = \frac{a}{ ax }(a^2 x^2 - 1)^{-1/2}$	(2.315)	$\frac{d}{dx}(\operatorname{arccot} ax) = -a(a^2 x^2 + 1)^{-1}$	(2.316)

^a a is a constant.

Hyperbolic derivatives^a

$\frac{d}{dx}(\sinh ax) = a \cosh ax$	(2.317)	$\frac{d}{dx}(\cosh ax) = a \sinh ax$	(2.318)
$\frac{d}{dx}(\tanh ax) = a \operatorname{sech}^2 ax$	(2.319)	$\frac{d}{dx}(\operatorname{csch} ax) = -a \operatorname{csch} ax \cdot \coth ax$	(2.320)
$\frac{d}{dx}(\operatorname{sech} ax) = -a \operatorname{sech} ax \cdot \tanh ax$	(2.321)	$\frac{d}{dx}(\coth ax) = -a \operatorname{csch}^2 ax$	(2.322)
$\frac{d}{dx}(\operatorname{arsinh} ax) = a(a^2 x^2 + 1)^{-1/2}$	(2.323)	$\frac{d}{dx}(\operatorname{arcosh} ax) = a(a^2 x^2 - 1)^{-1/2}$	(2.324)
$\frac{d}{dx}(\operatorname{artanh} ax) = a(1 - a^2 x^2)^{-1}$	(2.325)	$\frac{d}{dx}(\operatorname{arcsch} ax) = -\frac{a}{ ax }(1 + a^2 x^2)^{-1/2}$	(2.326)
$\frac{d}{dx}(\operatorname{arsech} ax) = -\frac{a}{ ax }(1 - a^2 x^2)^{-1/2}$	(2.327)	$\frac{d}{dx}(\operatorname{arcoth} ax) = a(1 - a^2 x^2)^{-1}$	(2.328)

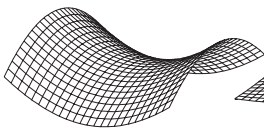
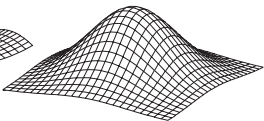
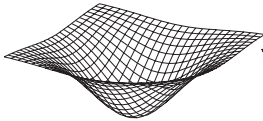
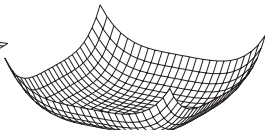
^a a is a constant.



Partial derivatives

Total differential	$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$	(2.329)	f $f(x, y, z)$
Reciprocity	$\left. \frac{\partial g}{\partial x} \right _y \left. \frac{\partial x}{\partial y} \right _g \left. \frac{\partial y}{\partial g} \right _x = -1$	(2.330)	g $g(x, y)$
Chain rule	$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial u}$	(2.331)	
Jacobian	$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$	(2.332)	J Jacobian u $u(x, y, z)$ v $v(x, y, z)$ w $w(x, y, z)$
Change of variable	$\int_V f(x, y, z) dx dy dz = \int_{V'} f(u, v, w) J du dv dw$	(2.333)	V volume in (x, y, z) V' volume in (u, v, w) mapped to by V
Euler–Lagrange equation	if $I = \int_a^b F(x, y, y') dx$ then $\delta I = 0$ when $\frac{\partial F}{\partial y} = \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right)$	(2.334)	y' dy/dx a, b fixed end points

Stationary points^a

			
saddle point	maximum	minimum	quartic minimum
Stationary point if $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ at (x_0, y_0) (necessary condition)			
Additional sufficient conditions			
for maximum	$\frac{\partial^2 f}{\partial x^2} < 0$, and $\frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} > \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2$		
for minimum	$\frac{\partial^2 f}{\partial x^2} > 0$, and $\frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} > \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2$		
for saddle point	$\frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} < \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2$		

^aOf a function $f(x, y)$ at the point (x_0, y_0) . Note that at, for example, a *quartic minimum* $\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = 0$.



Differential equations

Laplace	$\nabla^2 f = 0$	(2.339)	f	$f(x, y, z)$
Diffusion ^a	$\frac{\partial f}{\partial t} = D \nabla^2 f$	(2.340)	D	diffusion coefficient
Helmholtz	$\nabla^2 f + \alpha^2 f = 0$	(2.341)	α	constant
Wave	$\nabla^2 f = \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2}$	(2.342)	c	wave speed
Legendre	$\frac{d}{dx} \left[(1-x^2) \frac{dy}{dx} \right] + l(l+1)y = 0$	(2.343)	l	integer
Associated Legendre	$\frac{d}{dx} \left[(1-x^2) \frac{dy}{dx} \right] + \left[l(l+1) - \frac{m^2}{1-x^2} \right] y = 0$	(2.344)	m	integer
Bessel	$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - m^2)y = 0$	(2.345)		
Hermite	$\frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + 2\alpha y = 0$	(2.346)		
Laguerre	$x \frac{d^2 y}{dx^2} + (1-x) \frac{dy}{dx} + \alpha y = 0$	(2.347)		
Associated Laguerre	$x \frac{d^2 y}{dx^2} + (1+k-x) \frac{dy}{dx} + \alpha y = 0$	(2.348)	k	integer
Chebyshev	$(1-x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + n^2 y = 0$	(2.349)	n	integer
Euler (or Cauchy)	$x^2 \frac{d^2 y}{dx^2} + ax \frac{dy}{dx} + by = f(x)$	(2.350)	a, b	constants
Bernoulli	$\frac{dy}{dx} + p(x)y = q(x)y^a$	(2.351)	p, q	functions of x
Airy	$\frac{d^2 y}{dx^2} = xy$	(2.352)		

^aAlso known as the “conduction equation.” For thermal conduction, $f \equiv T$ and D , the thermal diffusivity, $\equiv \kappa \equiv \lambda/(\rho c_p)$, where T is the temperature distribution, λ the thermal conductivity, ρ the density, and c_p the specific heat capacity of the material.

